Day I : distributed parameters systems modelling - the port Hamiltonian approach

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Sketch of the talk

Motivation for the port-Hamiltonian approach

General motivation for the port-Hamiltonian approach

- practical solutions for control laws remain hard to achieve (solution of operators equations, infinite dimensional control)
- they are no general results for nonlinear DPS

⇒ *Particular cases*: **systems of conservation laws** with I/O or port variables

Use physical insight explicitly

- **modular modelling** (reusability, parallel computation)
- **integration schemes conserving physical invariants**
- **physically-based control design**, process and control co-design

Inspired from thermodynamics and mechanics

- **Irreversible thermodynamics**: systems of balance equations and phenomenological laws
- **Analytical mechanics**: Hamiltonian variational formulations in (q, p) coordinates

Finite dimensional Port-Controlled Hamiltonian (PCH) systems

Hamiltonian mechanics with port variables

Hamilton equations

$$\begin{pmatrix} \frac{\partial q}{\partial t} \\ \frac{\partial p}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p}(q,p) \\ -\frac{\partial H}{\partial q}(q,p) + \tau \end{pmatrix}$$

with q the generalized coordinates, p the corresponding momenta, τ the generalized forces and $H(q,p) = P(q) + \frac{1}{2}p^T M^{-1}(q)p$ the total energy (**Hamiltonian**) with $M(q)$ the inertia matrix and $P(q)$ the potential energy

Balance equation and passivity

$$\frac{dH}{dt} = \frac{\partial^T H}{\partial q}\dot{q} + \frac{\partial^T H}{\partial p}\dot{p} = \frac{\partial^T H}{\partial p}\tau = \dot{q}^T\tau = y^T(t)u(t)$$

with $y := \dot{q}$ and $u := \tau$. Therefore if $P(q)$ is bounded from below, the system is **passive** (conservative) w.r.t. the pair of **power-conjugated variables** (u, y) and the **storage function** $H(q,p)$

Conservative Port-Controlled Hamiltonian (PCH) systems

$$
\begin{aligned}
\begin{pmatrix} \frac{\partial q}{\partial t} \\ \frac{\partial p}{\partial t} \end{pmatrix} &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} \tau \\
v &= \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{pmatrix}
\end{aligned}
$$

Generalization in local state-space coordinates

$$
\begin{aligned}
\dot{x} &= J(x)\frac{\partial H}{\partial x} + g(x)u \\
y &= g^T(x)\frac{\partial H}{\partial x}
\end{aligned}
$$

where $u, y \in \mathbb{R}^m$ are power conjugated input-output variables and $x = (x_1, \ldots, x_n)$ are local coordinates for the n-dimensional state space manifold $\mathcal{X}$

Poisson structure

The interconnection structure $J(x)$ usually satisfies

1. skew-symmetry $J(x) = -J^T(x)$ which implies conservativeness and **passivity** w.r.t. (u, y) and the storage function H:

$$\frac{dH}{dt}(x(t)) = \frac{\partial^T H}{\partial x}\dot{x} = \frac{\partial^T H}{\partial x}\left(J(x)\frac{\partial H}{\partial x} + g(x)u\right) = y^T(t)u(t)$$

2. Jacobi identities

$$\sum_{l=1}^{n}\left[J_{lj}\frac{\partial J_{ik}}{\partial x_l} + J_{li}\frac{\partial J_{kj}}{\partial x_l} + J_{lk}\frac{\partial J_{ji}}{\partial x_l}\right] = 0 \ \forall i, j, k$$

which guarantee **integrability** and existence of **canonical local coordinates** $\tilde{x} = (q, p, s)$ s.t. $J(x)$ reduces to (**symplectic structure**)

$$\tilde{J} = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

PCH example: LCTG circuits

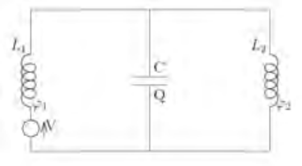


Figure: open (controlled) LC circuit

$$\begin{bmatrix} \dot{Q} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{J \text{ skew-symmetric and constant}} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial \varphi_1} \\ \frac{\partial H}{\partial \varphi_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t)$$

where

$$H(\varphi_1, \varphi_2, Q) = \frac{\varphi_1^2}{2L_1} + \frac{\varphi_2^2}{2L_2} + \frac{Q^2}{2C} \ ; \ u(t) := V(t) \ ; \ y(t) := \frac{\partial H}{\partial \varphi_1} = I_{\varphi_1}$$

PCH systems with dissipation

Terminating some ports with resistive elements

$$g(x)u = \begin{bmatrix} \tilde{g}(x) & g_R(x) \end{bmatrix} \begin{bmatrix} \tilde{u} \\ u_R \end{bmatrix}$$

$$\begin{bmatrix} \tilde{y} \\ y_R \end{bmatrix} = \begin{bmatrix} \tilde{g}^T(x) \\ g_R^T(x) \end{bmatrix} \frac{\partial H}{\partial x}(x)$$

$$u_R = -S(x) y_R \; ; \; S \geq 0$$

leads to the **input-state-output port-Hamiltonian** formulation:

$$\begin{aligned} \dot{x} &= (J(x) - R(x)) \frac{\partial H}{\partial x} + \tilde{g}(x)\tilde{u} \\ \tilde{y} &= \tilde{g}^T(x) \frac{\partial H}{\partial x} \end{aligned}$$

where $R(x) := g_R(x) S(x) g_R^T(x) \geq 0$ and

$$\frac{dH}{dt}(x(t)) = y^T(t)u(t) - \frac{\partial^T H}{\partial x} R(x) \frac{\partial H}{\partial x} \leq y^T(t)u(t)$$

PCH example with dissipation: capacitor microphone

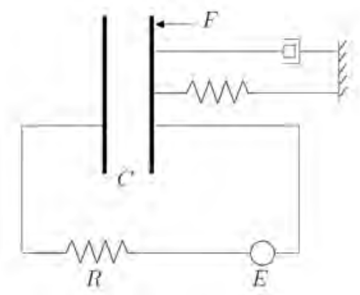


Figure: The capacitance $C(q)$ is varying with the displacement q of the right plate with mass m. This plate is attached to a linear spring with constant k and a linear damper with constant c. It is actuated by a mechanical force F (air pressure arising from sound). E is considered as a voltage source

Capacitor microphone example ... continued

The dynamical equations of motion can be written as the port-Hamiltonian system with dissipation

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{Q} \end{bmatrix} = \left(\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{J} - \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1/R \end{bmatrix}}_{R} \right) \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial Q} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/R \end{bmatrix}}_{\tilde{g}} \begin{bmatrix} F \\ E \end{bmatrix}$$

where

$$H(\varphi_1, \varphi_2, Q) = \frac{1}{2}k(q - q_e)^2 + \frac{p^2}{2m} + \frac{Q^2}{2C(q)}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial Q} \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p} \\ \frac{1}{R}\frac{\partial H}{\partial Q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ I \end{bmatrix}$$

Implicit Port Hamiltonian systems

- [Courant 1990] *Dirac manifolds*, Trans. American Math. Soc. 319:631–661
- [Dorfman 1993] *Dirac Structures and Integrability of Nonlinear Evolution Equations*, Wiley
- [Dalsmo & van der Schaft 1999] *On representations and integrability of mathematical structures in energy-conserving physical systems*, SIAM J. of Contr. and Opt., 37(1):54–91, 1999
- [Cervera & al. 2007] *Interconnection of port-Hamiltonian systems and composition of Dirac structures*, Automatica, 43:212-225

The bond space of power conjugated effort and flow variables

Let $\mathcal{F}$ and $\mathcal{E}$ be two real vector spaces and assume that they are endowed with a non degenerated bilinear form, called pairing denoted by:

$$\begin{array}{rcl} \langle .|\,.\rangle : \mathcal{F} \times \mathcal{E} & \to & \mathbb{R} \\ (f,\, e) & \mapsto & \langle e|\, f\rangle \end{array}$$

On the product space, called bond space:

$$\mathcal{B} = \mathcal{F} \times \mathcal{E}$$

the bilinear product leads to the definition of a symmetric bilinear form, called plus pairing as follows:

$$\begin{array}{rcl} \ll \cdot, \cdot \gg : \mathcal{B} \times \mathcal{B} & \to & \mathbb{R} \\ ((f_1,\, e_1)\,,\, (f_{,2}\; e_2)) & \mapsto & \ll (f_1, e_1), (f_2, e_2) \gg := \langle e_1|\, f_2\rangle + \langle e_2|\, f_1\rangle \end{array}$$

Dirac structure on vector spaces

Definition

A Dirac structure is a linear subspace $\mathcal{D} \subset \mathcal{B}$ such that $\mathcal{D} = \mathcal{D}^{\perp}$, with $\perp$ denoting the orthogonal complement with respect to the bilinear form $\ll, \gg$.

If a linear subspace $\mathcal{D} \subset \mathcal{B}$ satisfies only the isotropy condition $\mathcal{D} \subset \mathcal{D}^{\perp}$, what means that it is not maximal, one say that it is a *Tellegen structure*.

This name arises from the fact that the condition $\mathcal{D} \subset \mathcal{D}^{\perp}$ is equivalent to: $\langle e | f \rangle = 0, \forall (f, e) \in \mathcal{D}$.
This condition is known as Tellegen's theorem for the admissible voltages $e \in \mathbb{R}^n$ and currents $f \in \mathbb{R}^n$ of an electrical network , endowed with the Euclidean product.

Representations of finite-dimensional Dirac structures

Dirac structure admit algebraic definitions, based on some skew-symmetric linear maps called representation of a Dirac structure.

- [T.J. Courant. Dirac manifolds. Trans. American Math. Soc. 319, pages 631–661, 1990]
- [M. Dalsmo and A.J. van der Schaft. On representations and integrability of mathematical structures in energy-conserving physical systems. SIAM Journal of Control and Optimization, 37(1):54–91, 1999.]

Assume that the flow and effort spaces are finite-dimensional and choose

$$\mathcal{F} = \mathcal{E} = \mathbb{R}^n$$

with the power pairing being the canonical Euclidean product in $\mathbb{R}^n$ composed with a signature matrix σ:

$$\langle e | f \rangle = e^T \sigma f \quad \text{where } f \in \mathcal{F} = \mathbb{R}^n,\ e \in \mathcal{E} = \mathbb{R}^n$$

Representations of a Dirac structure ... continued

Theorem

A Dirac structure $\mathcal{D} \subset \mathcal{B} = \mathbb{R}^n \times \mathbb{R}^n$ admits two $n \times n$ real matrices, denoted here E and F, and satisfying

$$E\sigma F^T + F\sigma E^T = 0$$

$$rank\, [E : F] = n$$

$\mathcal{D}$ admits the image representation:

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{E} | f = E^T\lambda,\ e = F^T\lambda,\ \lambda \in \mathbb{R}^n\}$$

and $\mathcal{D}$ also admits the kernel representation:

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{E} | \,(F\,\sigma)\ f + (E\,\sigma)\ e = 0\}$$

Representations of a Dirac structure ... continued

Theorem

A Dirac structure $\mathcal{D} \subset \mathcal{B} = \mathbb{R}^n \times \mathbb{R}^n$ admits a decomposition of the flow and effort spaces:

$$\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \ni \begin{pmatrix} f_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ f_2 \end{pmatrix} \text{ and } \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \ni \begin{pmatrix} e_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ e_2 \end{pmatrix}$$

and a skew-symmetric $n \times n$ real matrix denoted J which define the input-output representation:

$$\mathcal{D} = \{(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}) \in \mathcal{F} \times \mathcal{E} | \begin{pmatrix} f_1 \\ e_2 \end{pmatrix} = J \begin{pmatrix} e_1 \\ f_2 \end{pmatrix}\}$$

Hamiltonian system defined w.r.t. a Dirac structure

We consider the case where the state space is a real vector space $\mathcal{F}$ of dimension n. Then the flow space is again $\mathcal{F}$ and the effort space will be chosen as its dual space $\mathcal{F}^*$. Then the bond space is simply $\mathcal{F} \times \mathcal{F}^*$. The power pairing is defined using the duality product:

$$P(t) = \langle f \,|e \rangle := \langle e, f \rangle \; ; \; \forall (f, e) \in \mathcal{F} \times \mathcal{F}^*$$

Definition

Consider a Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$. A *Hamiltonian system* with respect to the Dirac structure $\mathcal{D}$ generated by the Hamiltonian function $H \in C^\infty(\mathcal{F}, \mathbb{R})$, is defined by the implicit differential equation: $\left(\frac{dx}{dt}, \frac{\partial H}{\partial x}\right) \in \mathcal{D}$.

The isotropy of the Dirac structure implies the conservation of the Hamiltonian:

$$\frac{dH}{dt} = \left\langle \frac{dx}{dt} \Big| \frac{\partial H}{\partial x} \right\rangle = 0$$

Example: closed LCTG circuits

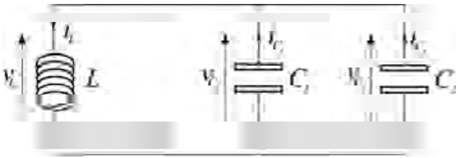
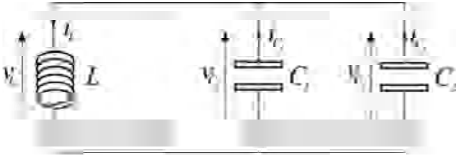


Figure: A simple LC circuit composed of two capacitors and an inductor

Kirchhoff's laws define the kernel representation of a Dirac structure:

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{F} \begin{pmatrix} i_{C_1} \\ v_L \\ i_{C_2} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}}_{E} \begin{pmatrix} v_{C_1} \\ i_L \\ v_{C_2} \end{pmatrix} = 0$$

Example: closed LCTG circuit ... continued



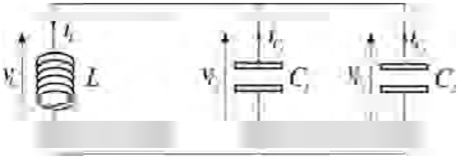
The chosen state variables are:

$$x = \begin{pmatrix} q_1 \\ \phi \\ q_2 \end{pmatrix} \begin{matrix} \text{charge of capacitor } 1 \\ \text{flux in inductor} \\ \text{charge of capacitor } 2 \end{matrix} \quad \textit{then } \frac{dx}{dt} = \begin{pmatrix} i_{C_1} \\ v_L \\ i_{C_2} \end{pmatrix} \in \mathcal{F}$$

The conjugated variables are the derivatives of the total electro-magnetic energy : $H(x)$ w.r.t. the state variables, i.e.:

$$\frac{\partial H}{\partial x}(x) = \begin{pmatrix} v_{C_1} \\ i_L, \\ v_{C_2} \end{pmatrix} \in \mathbb{R}^3 = \mathcal{E}$$

Example: closed LCTG circuit ... continued



The dynamics of the LC circuit is an implicit Hamiltonian system.

$$\left(\frac{dx}{dt}, \frac{\partial H}{\partial x}\right) \in \mathcal{D} \Leftrightarrow \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{F} \frac{d}{dt}\begin{pmatrix} q_1 \\ \phi \\ q_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}}_{E} \frac{\partial H}{\partial x}(x) = 0$$

The rank degeneracy of F corresponds to the constraint:

$$\frac{\partial H}{\partial q_1}(x) - \frac{\partial H}{\partial q_2}(x) = 0$$

Extension to Port Hamiltonian Systems

Tangent and co-tangent spaces of the state space are augmented with a set of port variables used to represent the ***interaction*** *of the system with its* ***environment***

We augment the bond space according to

$$\mathcal{B} := \left\{ \left(\left(\begin{array}{c} f^i \\ f^e \end{array} \right), \left(\begin{array}{c} e^i \\ e^e \end{array} \right) \right) \in \mathcal{F} \times \mathcal{E} = \left(\mathcal{F}^i \times \mathcal{F}^e \right) \times \left(\mathcal{F}^{i*} \times \mathcal{E}^e \right) \right\}$$

with the help of two other finite-dimensional port spaces $\mathcal{F}^e$ and $\mathcal{E}^e$ endowed with the non degenerated bilinear form $\langle .|\, .\rangle_e$.
The augmented bond space $\mathcal{B}$ is then endowed with the bilinear form:

$$\left\langle \left(\begin{array}{c} f^i \\ f^e \end{array} \right) \middle| \left(\begin{array}{c} e^i \\ e^e \end{array} \right) \right\rangle = \left\langle e^i \middle| f^i \right\rangle_i + \left\langle e^e \middle| f^e \right\rangle_e$$

Port Hamiltonian Systems

Definition

A **port Hamiltonian system** w.r.t. the Dirac structure $\mathcal{D} \subset \mathcal{B}$ and generated by the Hamiltonian function, $H \in C^{\infty}\left(\mathcal{F}^i, \mathbb{R}\right)$ is defined by the implicit differential equation:

$$\left(\begin{pmatrix} \frac{dx}{dt} \\ f^e \end{pmatrix}, \begin{pmatrix} \frac{\partial H}{\partial x} \\ e^e \end{pmatrix}\right) \in \mathcal{D}$$

- Port Hamiltonian systems do not satisfy the Cauchy conditions as long as the system is not completed with some relations on the external port variables (f^e, e^e) .
- Complex systems composed of a set of interacting subsystems may be described using the Composition of Dirac structures rather than interaction Hamiltonian functions
- The isotropy property of the Dirac structure translates now into a balance equation for the Hamiltonian:

$$0 = \left\langle \frac{\partial H}{\partial x} \middle| \frac{dx}{dt} \right\rangle + \left\langle e^e \middle| f^e \right\rangle_e = \frac{dH}{dt} + \left\langle e^e \middle| f^e \right\rangle_e$$

Example: open LCTG circuit

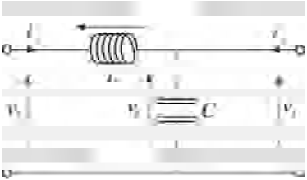


Figure: An open LC circuit with two external ports

Kirchhoff's laws define the kernel representation of an extended Dirac structure

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{F} \begin{pmatrix} i_C \\ v_L \\ i_1 \\ i_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}}_{E} \begin{pmatrix} v_C \\ i_L \\ v_1 \\ v_2 \end{pmatrix} = 0$$

Open LCTG circuit ... continued

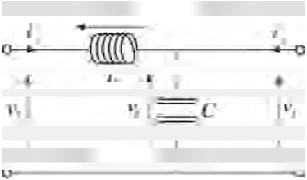


Figure: An open LC circuit with two external ports

The chosen state variables are:

$$x = \begin{pmatrix} q \\ \phi \end{pmatrix} \begin{array}{l} \text{charge of capacitor} \\ \text{flux in inductor} \end{array} \Rightarrow \frac{dx}{dt} = \begin{pmatrix} i_{C_1} \\ v_L \end{pmatrix} \in \mathcal{F}^i$$

The conjugated variables are the derivatives of the total electro-magnetic energy $H(x)$ w.r.t. the state variables, i.e.:

$$\frac{\partial H}{\partial x}(x) = \begin{pmatrix} v_C \\ i_L, \end{pmatrix} \in \mathbb{R}^2 = \mathcal{F}^{i*}$$

Open LCTG circuit ... continued

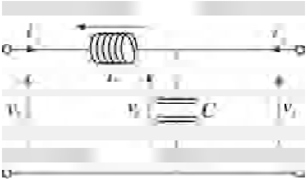


Figure: An open LC circuit with two external ports

The two open ports define the vector spaces of external currents $\mathcal{F}^e = \mathbb{R}^2 \ni (i_1, i_2)$ and voltages $\mathcal{E}^e = \mathbb{R}^2 \ni (v_1, v_2)$. Then the Dirac structure defined with the usual euclidian product is exactly

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{F} \begin{pmatrix} \frac{dq}{dt} \\ \frac{d\phi}{dt} \\ i_1 \\ i_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}}_{E} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial \phi} \\ v_1 \\ v_2 \end{pmatrix} = 0$$

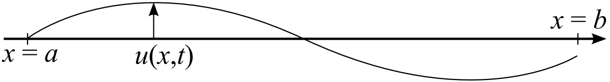
Partial conclusion on implicit PCH systems

Port Hamiltonian systems are an extension of Hamiltonian systems which:

- are defined by extending the internal spaces of tangent and cotangent spaces by a pair of conjugated port vector spaces
- are using a Dirac structure on these extended effort and flow spaces
- are similar to (constrained) Poisson Hamiltonian systems
- allows to write balance equations including energy flows from the environment (hence to prove passivity properties)
- allows interconnection (including feedback control!) through composition of Dirac structures

Hyperbolic systems of two conservation laws: the string and the Shallow Water examples

The linear vibrating string example



Classical mechanics state variables are:

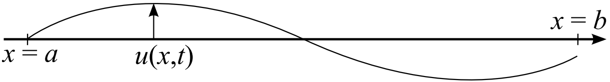
$$u(x,t) \textit{ string vertical displacement}$$
$$v(x,t) = \frac{\partial u(x,t)}{\partial t} \textit{ string vertical velocity}$$

The Newton and Hooke laws (restoring force proportional to the deformation slope) lead to:

$$\frac{\partial}{\partial t}\left(\mu(x)v(x,t)\right) = \mu(x)\frac{\partial^2 u(x,t)}{\partial t^2} = -\frac{\partial}{\partial x}\left(-T(x)\frac{\partial u(x,t)}{\partial x}\right)$$

where $\mu(x)$ is the lineic mass density and $T(x)$ the elastic modulus

Linear spring ... continued



Choosing the **energy state variables** $z(x,t) := (\varepsilon, p)^T$ with:

$$\begin{aligned}\varepsilon(x,t) &= \frac{\partial u(x,t)}{\partial x} && \textit{strain}\\ p(x,t) &= \mu(x) v(x,t) && \textit{momentum}\end{aligned}$$

The **total energy** of the string is

$$H(z,t) = \int_a^b \frac{1}{2} z^T(x,t) \begin{pmatrix} T(x) & 0 \\ 0 & \frac{1}{\mu(x)} \end{pmatrix} z(x,t)\, dx$$

and only depends on the state variables (and not on their spatial derivatives)

Linear spring ... continued

variational derivative

Consider a total energy functional

$$H[z] = \int_a^b \mathcal{H}\left(x,\, z,\, z^{(1)}, \ldots, x^{(n)}\right) dx$$

where the energy density $\mathcal{H}$ is a smooth function, the ***variational derivative*** $\frac{\delta H}{\delta z}$ is defined such that:

$$H[z + \epsilon\delta z] = H[z] + \epsilon \int_a^b \frac{\delta H}{\delta z}\, \delta z\, dx + O(\epsilon^2)$$

for every $\epsilon \in \mathbb{R}$ and every smooth real function $\delta z(x)$ such that : $\delta z^{(i)}(a) = \delta z^{(i)}(b) = 0,\ i = 0, \ldots, n$

Linear spring ... continued

When $\mathcal{H}$ only depends on x (and not its derivatives) and when the integration domain is fixed:

$$\frac{\delta H}{\delta z} = \frac{\partial \mathcal{H}}{\partial z}$$

Example

In the linear spring example, the variational derivatives of H w.r.t. $\varepsilon(x,t)$ and $p(x,t)$ are the **co-energy** or **distributed effort** variables:

$$\begin{aligned}
\frac{\delta H}{\delta \varepsilon} &= T(x)\varepsilon(x,t) = \sigma(x,t) \qquad \textit{stress} \\
\frac{\delta H}{\delta p} &= \frac{p(x,t)}{\mu(x)} = v(x,t) \qquad \textit{velocity}
\end{aligned}$$

Linear spring ... continued

Hamiltonian formulation for the linear spring

The classical model for the linear spring may then be expressed as the infinite-dimensional Hamiltonian system [Olver 1993] of two conservation laws:

$$\underbrace{\frac{\partial}{\partial t}\begin{bmatrix} \varepsilon(x,t) \\ p(x,t) \end{bmatrix}}_{\textit{flows}} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} \sigma(x,t) \\ v(x,t) \end{bmatrix}$$

$$= -\frac{\partial}{\partial x} \quad \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \delta_\varepsilon H(\varepsilon,p) \\ \delta_p H(\varepsilon,p) \end{bmatrix}}$$

defining the **flux vector** $\begin{bmatrix} v(x,t) \\ \sigma(x,t) \end{bmatrix}$

Linear spring ... continued

Energy balance equation

The canonical Hamiltonian operator [Morrison 1980]

$$\mathcal{J} = \begin{pmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{pmatrix}$$

is **formally skew-symmetric**:

$$\begin{aligned} \frac{dH}{dt} &= \int_a^b \frac{\delta H^T}{\delta z} \dot{z}(x,t)\, dx \\ &= \int_a^b \left[\begin{array}{cc} \sigma(x,t) & v(x,t) \end{array} \right] \left[\begin{array}{cc} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{array} \right] \left[\begin{array}{c} \sigma(x,t) \\ v(x,t) \end{array} \right] dx \\ &= \int_a^b \frac{\partial}{\partial x} \left[\sigma(x,t) v(x,t) \right] dx \\ &= -\sigma(a,t) v(a,t) + \sigma(b,t) v(b,t) \\ &= 0 \end{aligned}$$

for zero boundary values of the co-energy (effort) variables

Linear spring ... continued

Extending the system with boundary port variables

The energy balance equation

$$\frac{dH}{dt} = -\sigma(a,t)v(a,t) + \sigma(b,t)v(b,t)$$

suggests the definition of external **boundary port variables**

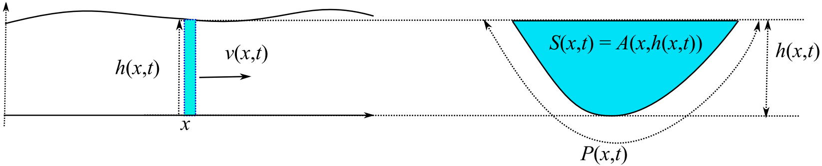
$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} := \begin{bmatrix} \delta_\varepsilon H(\varepsilon, p) \\ \delta_p H(\varepsilon, p) \end{bmatrix}\Bigg|_{x\in\{a,b\}} = \begin{bmatrix} \sigma(x,t) \\ v(x,t) \end{bmatrix}\Bigg|_{x\in\{a,b\}}$$

which complete the system of two conservation laws and allow to define a **port-Hamiltonian system** which is trivially **passive w.r.t. power conjugated input-output** choices $(u(t), y(t)) \in \mathbb{R}^2 \times \mathbb{R}^2$, i.e.

$$\frac{dH}{dt} = f_\partial^b e_\partial^b - f_\partial^a e_\partial^a =: y^T(t)u(t)$$

The nonlinear shallow water equations (SWE) example

[Hamroun et al. 2006] Trans. Fluid Mechanics 1(12), 995-1009



State (energy) variables are chosen as **differential forms**:

$$\begin{aligned} q(x,t) &= \rho S(x,t)dx \quad & \textit{mass density} \\ p(x,t) &= \rho v(x,t)dx \quad & \textit{momentum density} \end{aligned}$$

The energy density (non quadratic and non separable) may be written in this SWE example

$$\begin{aligned} \mathcal{H}(x,t) &= \rho\left(g\left(hA(x,h) - \int_0^h A(x,\xi)d\xi\right) + \frac{S(x,t)v^2(x,t)}{2}\right)dx \\ &= \mathcal{H}(q,p) \end{aligned}$$

The shallow water equations ... continued

Conservation equations

The simplified Saint-Venant equations are a system of 2 conservation laws

$$\frac{\partial q}{\partial t} = -\frac{\partial}{\partial z}\left(S(x,t)v(x,t)\right)\, dx \quad \textbf{mass}$$

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial z}\left(\rho\left(gh(x,t)+\frac{v^2(x,t)}{2}\right)\right)\, dx \quad \textbf{momentum}$$

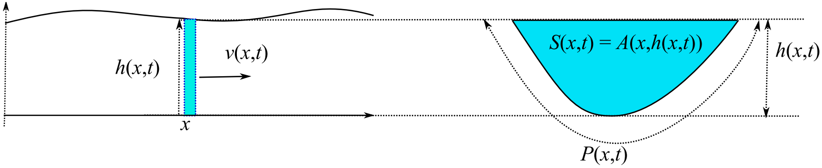
Flow and effort variables

Flow variables could be defined as the differential forms $(\dot{q}, \dot{p})$.
Effort variables are variational derivatives of H w.r.t. $q(x,t)$ and $p(x,t)$:

$$e_p = \delta_p H = \frac{\partial \mathcal{H}}{\partial p} = S(x,t)v(x,t) \quad \textbf{water flow}$$

$$e_q = \delta_q H = \frac{\partial \mathcal{H}}{\partial q} = \rho\left(gh(x,t)+\frac{v^2(x,t)}{2}\right) \quad \textbf{hydrodynamic pressure}$$

The shallow water equations ... continued



Hamiltonian formulation for the SWE

The SW system of two conservation laws may be written using the same canonical Hamiltonian operator as before relating the power conjugated **flow** $(f_q, f_p) := -(\dot{q}, \dot{p})$ and **effort** $(e_q, e_p) = (\delta_q H, \delta_p H)$ variables:

$$-\frac{\partial}{\partial t}\begin{bmatrix} q(x,t) \\ p(x,t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} \delta_q H(q,p) \\ \delta_p H(q,p) \end{bmatrix}$$

The shallow water equations ... continued

Power balance equation

$$
\begin{aligned}
\frac{dH}{dt} &= \frac{d}{dt}\int_Z \mathcal{H}(q,p) \\
&= \int_Z \delta_q H \wedge \dot{q} + \delta_p H \wedge \dot{p} \\
&= \int_Z e_q \wedge f_q + e_p \wedge f_p \\
&= -\int_Z e_q \wedge d\left(e_p\right) + e_p \wedge d\left(e_q\right) \\
&= -\int_Z d\left(e_q \wedge e_p\right) \\
&= -\int_{\partial Z} e_\partial \wedge f_\partial \\
&= p_d(0,t)Q(0,t) - p_d(L,t)Q(L,t)
\end{aligned}
$$

where $e_\partial := e_q|_{\partial Z}$ and $f_\partial := e_p|_{\partial Z}$

The shallow water equations ... continued

In the case of a reach with a bottom line $z_f(x)$ (i.e. a **slope** $\frac{dz_f}{dx}$) and **friction forces** $J(Q,h)$, the (full) shallow water model may still be written in the form of mass and momentum balance equations:

$$-\frac{\partial}{\partial t}\begin{bmatrix} q(x,t) \\ p(x,t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} \delta_q \tilde{\mathcal{H}}(q,p) \\ \delta_p \tilde{\mathcal{H}}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ f_d \end{bmatrix}$$

with the **modified energy density**

$$\tilde{\mathcal{H}}(q,p) = \mathcal{H} + \rho g z_f(x) S(x,t) dx = \mathcal{H} + g z_f(x) q$$

and the **dissipative flow** $f_d = \rho g J(Q,h) dx$.

The power balance equation becomes

$$\begin{aligned} \frac{d\tilde{H}}{dt} &= \int_{\partial Z} \tilde{e}_\partial \wedge f_\partial - \int_Z e_d \wedge f_d \\ &= \tilde{p}_d(0,t)Q(0,t) - \tilde{p}_d(L,t)Q(L,t) - \int_{[0,L]} \rho g Q J(Q,h) dx \end{aligned}$$

Observations from the string and SWE examples

The canonical Hamiltonian operator

The matrix differential operator

$$\mathcal{J} = \begin{pmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{pmatrix}$$

satisfies:

- skew-symmetry. Indeed, consider two vectors of smooth functions $e = (e_1, e_2)$ and $e' = (e'_1, e'_2)$ satisfying the **homogeneous boundary conditions** $(e_1 e_2)\,|_\partial = (e'_1 e'_2)\,|_\partial = 0$, then:

$$\begin{aligned} \int_a^b \left(e^t\, \mathcal{J}\, e' + e'^t\, \mathcal{J}\, e\right) dz &= \int_a^b \left[e_1 \left(\tfrac{\partial}{\partial z} e'_2\right) + e_2 \left(\tfrac{\partial}{\partial z} e'_1\right) \right. \\ &\quad \left. + e_1 \left(\tfrac{\partial}{\partial z} e'_2\right) + e_2 \left(\tfrac{\partial}{\partial z} e'_1\right)\right] dz \\ &= \left[e_1\, e'_2 + e_2\, e'_1\right]_a^b \\ &= 0 \end{aligned}$$

- the Jacobi identities. Indeed, $\mathcal{J}$ is a constant coefficient differential operator

Observations from the examples... continued

Boundary Port Hamiltonian systems

For hyperbolic systems of conservation laws:

- Hamiltonian formulation makes appearing a **canonical Poisson bracket**
- Stokes theorem leads to the definition of **power-conjugated boundary variables**
- This suggests the extension of the canonical Poisson structure to a **Stokes-Dirac** structure (including boundary port variables)

The definition of a class of boundary Port-Hamiltonian distributed parameters systems

[van der Schaft & Maschke 2002] *Hamiltonian formulation of distributed parameter systems with boundary energy flow*, J. of Geometry and Physics, 42:166-174

Canonical system of two conservation laws

Energy density

We consider a system with two coupled energy domains with energy

$$H(q,p) = \int_Z \mathcal{H}(q,p,z)$$

where Z is the spatial domain and $\mathcal{H} \in \Lambda^n(Z)$ the energy density:

$$\mathcal{H} : \Lambda^{n_p}(Z) \times \Lambda^{n_q}(Z) \times Z \rightarrow \Lambda^n(Z) : (p,q,z) \mapsto \mathcal{H}(p,q,z)$$

Power balance

$$\frac{dH}{dt} = \int_Z \left[\delta_q \mathcal{H} \wedge \frac{\partial q}{\partial t} + \delta_p \mathcal{H} \wedge \frac{\partial p}{\partial t} \right]$$

where $\delta_q \mathcal{H} \in \Lambda^{n-n_q}(Z)$ et $\delta_p \mathcal{H} \in \Lambda^{n-n_q}(Z)$ are variational derivatives (first order variations) of $\mathcal{H}$ w.r.t. q and p.

System of two conservation laws ... continued

Definition

The canonical Port-Hamiltonian form for a system of two conservation laws in the n-dimensional spatial domain Z whose energy variables $p \in \Lambda^{n_p}(Z)$ and $q \in \Lambda^{n_q}(Z)$ satisfy $n_p + n_q = n + 1$ and whose energy density is $\mathcal{H}(p, q) \in \Lambda^n(Z)$, is given by the following state space realization:

$$\begin{bmatrix} -\frac{\partial q}{\partial t} \\ -\frac{\partial p}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} \delta_q \mathcal{H} \\ \delta_p \mathcal{H} \end{bmatrix}$$

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-n_q} \end{bmatrix} \begin{bmatrix} \delta_q \mathcal{H}_{|\partial Z} \\ \delta_p \mathcal{H}_{|\partial Z} \end{bmatrix}$$

where $r = n_q \times n_p + 1$ and (f_∂, e_∂) are the boundary power conjugated port variables.

System of two conservation laws ... continued

Power balance

Key arguments are the anti-derivation formula and Stokes theorem

$$\begin{aligned}
\frac{dH}{dt} &= \int_Z [\delta_q \mathcal{H} \wedge \frac{\partial q}{\partial t} + \delta_p \mathcal{H} \wedge \frac{\partial p}{\partial t}] \\
&= \int_Z [\delta_q \mathcal{H} \wedge (-1)^r d(\delta_p \mathcal{H}) + \delta_p \mathcal{H} \wedge d(\delta_q \mathcal{H})] \\
&= \int_Z d\left(\delta_p \mathcal{H} \wedge \delta_q \mathcal{H}\right) \\
&= \int_{\partial Z} \delta_p \mathcal{H} \wedge \delta_q \mathcal{H} \\
&= \int_{\partial Z} f_\partial \wedge e_\partial
\end{aligned}$$

Instantaneous power conservation

$$-\int_Z [\delta_q \mathcal{H} \wedge \frac{\partial q}{\partial t} + \delta_p \mathcal{H} \wedge \frac{\partial p}{\partial t}] + \int_{\partial Z} f_\partial \wedge e_\partial = 0$$

System of two conservation laws ... continued

Remind the Dirac structure definition

Consider two linear spaces $\mathcal{F}$ and $\mathcal{E}$ endowed with the inner product $\ll \cdot, \cdot \gg$. The corresponding Dirac structure is the linear subspace $D \subset \mathcal{F} \times \mathcal{E}$ such that

$$D = D^{\perp}$$

Alternatively D is the graph of a linear skew-symmetric map $\mathcal{F} \to \mathcal{E}$.

System of two conservation laws ... continued

Conjugated port variables

We look for an implicit representation: the symplectic Port-Hamiltonian system is defined with the belonging of the power conjugated port variables

$$(e_q, e_p, e_\partial, f_q, f_p, f_\partial) := \left(\delta_q \mathcal{H}, \delta_p \mathcal{H}, (-1)^{n+1-n_q} \delta_p \mathcal{H}|_{\partial Z}, -\dot{q}, -\dot{p}, \delta_q \mathcal{H}|_{\partial Z}\right)$$

to a subspace of the bond space:

$$\mathcal{E} \times \mathcal{F} := \Lambda^{n-n_q}(Z) \times \Lambda^{n-n_p}(Z) \times \Lambda^{n-n_p}(\partial Z) \times \Lambda^{n_q}(Z) \times \Lambda^{n_p}(Z) \times \Lambda^{n-n_q}(\partial Z)$$

Power (plus) pairing

The natural power pairing $\langle \cdot | \cdot \rangle$ on $\mathcal{E} \times \mathcal{F}$ is defined as:

$$\begin{aligned} \langle \cdot | \cdot \rangle : \quad & \mathcal{E} \times \mathcal{F} \to \mathbb{R} \\ & (e, f) \mapsto \langle e | f \rangle := \int_Z e_q \wedge f_q + e_p \wedge f_p + \int_{\partial Z} e_\partial \wedge f_\partial \end{aligned}$$

System of two conservation laws ... continued

Inner product

The associated inner product on $\mathcal{E} \times \mathcal{F}$ is defined by symmetrization of the power pairing according to:

$$\begin{aligned} \langle\langle \cdot | \cdot \rangle\rangle : \quad & (\mathcal{E} \times \mathcal{F}) \times (\mathcal{E} \times \mathcal{F}) \to \mathbb{R} : \\ & ((e_1, f_1), (e_2, f_2)) \mapsto \langle\langle (e_1, f_1) | (e_2, f_2) \rangle\rangle := \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle \end{aligned}$$

Theorem: implicit representation of Stokes-Dirac structures

Let $\mathcal{D}$ be the linear subspace of the bond space $\mathcal{E} \times \mathcal{F}$ given in (46,1), i.e.:

$$\mathcal{D} := \left\{ (e_q, e_p, e_\partial, f_q, f_p, f_\partial) \in \mathcal{E} \times \mathcal{F} \middle| \begin{bmatrix} f_q \\ f_p \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_q \\ e_p \end{bmatrix} \right. \\ \left. \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^{n+1-n_q} \end{bmatrix} \begin{bmatrix} e_q|_{\partial Z} \\ e_p|_{\partial Z} \end{bmatrix} \right\}$$

Then $\mathcal{D} = \mathcal{D}^\perp$. $\mathcal{D}$ is uniquely defined as the Dirac structure $\mathcal{D} \subset \mathcal{B} := \mathcal{E} \times \mathcal{F}$ associated to the inner product $\langle\langle \cdot | \cdot \rangle\rangle$.

System of two conservation laws ... continued

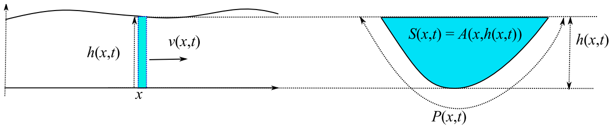
Remark

$$\left(\delta_q\mathcal{H}, \delta_p\mathcal{H}, (-1)^{n+1-n_q}\delta_p\mathcal{H}|_{\partial Z}, -\dot{q}, -\dot{p}, \delta_q\mathcal{H}|_{\partial Z}\right) \in \mathcal{D}$$

The isotropy condition $\mathcal{D} \subset \mathcal{D}^{\perp}$ implies power conservation:

$$(f, e) \in \mathcal{D} \Rightarrow < f|e >= 0$$

The Shallow water example (1/5)



The energy state variables are

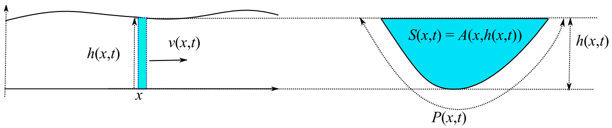
$$\begin{array}{rcll} q(x,t) & = & \rho S(x,t)dx & \textit{mass density} \\ p(x,t) & = & \rho v(x,t)dx & \textit{momentum density} \end{array}$$

The energy $H(t) := \int_a^b \mathcal{H}(x,t)\,dx$ is defined from the energy density

$$\begin{aligned} \mathcal{H}(x,t) \quad := \quad & \rho\left(g\left(hS(x,t) - \int_0^h A(x,\xi)d\xi\right) + \frac{S(x,t)v^2(x,t)}{2}\right)dx \\ = \quad & g\left(h(\star q)p - (\int_0^{h(\star q)} A(x,\xi)d\xi)\,dx\right) + \frac{\star q \star p}{2}p = \mathcal{H}(q,p) \end{aligned}$$

with $\star(\rho v(x,t)dx) = \star p = v(x,t)$ and $\star(\rho S(x,t)dx) = \star q = S(x,t)$.

The Shallow water example (2/5)



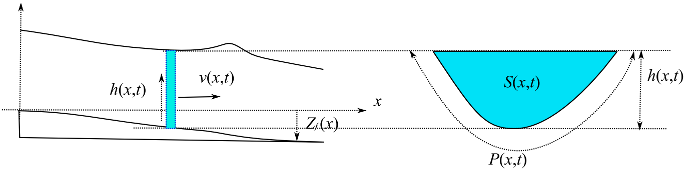
The BPHS corresponding to the SWE may be implicitely defined as

$$\left(\delta_q H, \delta_p H, -\delta_p H|_{\{a,b\}}, -\dot{q}, -\dot{p}, \delta_q H|_{\{a,b\}}\right) \in \mathcal{D}$$

with $\mathcal{D}$ being the 1D canonical Stokes-Dirac structure

$$\mathcal{D} := \left\{ (e_q, e_p, e_\partial, f_q, f_p, f_\partial) \in \mathcal{E} \times \mathcal{F} \middle| \begin{bmatrix} f_q \\ f_p \end{bmatrix} = \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_q \\ e_p \end{bmatrix} \right.$$
$$\left. \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e_q|_{\partial Z} \\ e_p|_{\partial Z} \end{bmatrix} \right\}$$

The Shallow water example (3/5)



In the case of a reach with a bottom line $z_f(x)$ and **friction forces**, the mass and momentum **balance** equations become:

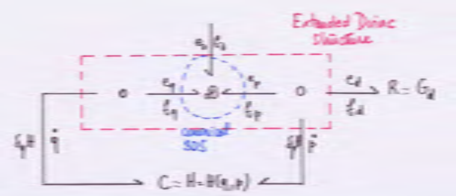
$$-\frac{\partial}{\partial t}\begin{bmatrix} q(x,t) \\ p(x,t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} \delta_q \tilde{\mathcal{H}}(q,p) \\ \delta_p \tilde{\mathcal{H}}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ f_d \end{bmatrix}$$

with the **modified energy density**

$$\tilde{\mathcal{H}}(q,p) = \mathcal{H} + g z_f(x) q$$

and the **dissipative flow** f_d.

The Shallow water example (4/5)



Extented Dirac structure

$$\begin{bmatrix} -\frac{\partial q}{\partial t} \\ -\frac{\partial p}{\partial t} \\ -e_d \end{bmatrix} = \begin{bmatrix} 0 & d & 0 \\ d & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} e_q \\ e_p \\ f_d \end{bmatrix} \text{ and } \begin{bmatrix} e_\partial \\ f_\partial \end{bmatrix} = \begin{bmatrix} -e_q|_\partial \\ e_p|_\partial \end{bmatrix}$$

$$\begin{aligned} e_q &= \delta_q \mathcal{H}(q,p) = \rho g \left(z_f + h(\star q)\right) + \frac{\star p}{2} p = \rho g \left(z_f(x) + h(x,t)\right) + \rho \frac{v^2(x,t)}{2} \\ e_p &= \delta_p \mathcal{H}(q,p) = \star p \star q = S(x,t) v(x,t) \\ f_d &= G_d \wedge e_p \end{aligned}$$

The Shallow water example (5/5)

Manning-Strickler friction formulae

The dissipation force $G_d = \rho(\star G_d)dx \in \Lambda^1(Z)$ may be computed using Manning-Strickler (or others) empirical formulae:

$$\star G_d = \frac{g\,|Q|}{K^2 S^2 R^{\frac{4}{3}}} = \frac{g\,|\star p|}{K^2\,|\star q|\,R(\star q)^{\frac{4}{3}}}$$

with $R := S/P$ (hydraulic radius).

Energy balance equation

Using Stokes theorem, the energy balance equation reads (in the case of a canal reach with slope and friction forces)

$$\begin{aligned} \frac{dH}{dt} &= \int_{\partial Z} e_\partial \wedge f_\partial - \int_Z e_d \wedge f_d = e_\partial^a f_\partial^a - e_\partial^b f_\partial^b - \int_{[a,b]} e_d \wedge f_d \\ &\leq e_\partial^a f_\partial^a - e_\partial^b f_\partial^b \end{aligned}$$

Port-Hamiltonian systems which are not systems of two conservation (balance) equations

Burger's equation (1/2)

Using the Hamiltonian function

$$H(v) := \int_0^L \mathcal{H}(v) dx \;\text{ with }\; \mathcal{H}(v) := \frac{v^3}{6}$$

The Burger's momentum **balance** equation may be written:

$$\frac{\partial v}{\partial t}(x,t) = -\frac{\partial}{\partial x}\left[\delta_v \mathcal{H}(v)\right] = -\frac{\partial}{\partial x}\left[\frac{v^2(x,t)}{2}\right]$$

And the energy balance equation reads

$$\begin{aligned}
\frac{d}{dt}H(v(t)) &= \int_0^L \delta_v \mathcal{H}(v)\frac{\partial v}{\partial t} dx = -\int_0^L \frac{\partial}{\partial x}\left(\frac{[\delta_v \mathcal{H}(v)]^2}{2}\right) dx \\
&= \frac{[\delta_v \mathcal{H}(v(0,t))]^2}{2} - \frac{[\delta_v \mathcal{H}(v(L,t))]^2}{2} \\
&= \underbrace{\left(\frac{\delta_v \mathcal{H}(v(0,t)) - \delta_v \mathcal{H}(v(L,t))}{\sqrt{2}}\right)}_{e_\partial} \underbrace{\left(\frac{\delta_v \mathcal{H}(v(0,t)) + \delta_v \mathcal{H}(v(L,t))}{\sqrt{2}}\right)}_{f_\partial}
\end{aligned}$$

Burger's equation (2/2)

The Burger's equation may be written

$$(\delta_v \mathcal{H}, e_\partial, \dot{v}, f_\partial) \in \mathcal{D} := \left\{ (e_v, e_\partial, f_v, f_\partial) \in \mathcal{E} \times \mathcal{F} \middle| f_v = -\frac{\partial e_v}{\partial x} \right.$$
$$\left. \begin{bmatrix} e_\partial \\ f_\partial \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} +1 & -1 \\ +1 & +1 \end{bmatrix} \begin{bmatrix} v_0^2/2 \\ v_L^2/2 \end{bmatrix} \right\}$$

where the Dirac structure is associated to the power pairing

$$\langle (e_v, e_\partial), (f_v, f_\partial) \rangle := \int_0^L e_v f_v dx - e_\partial f \partial$$

Heat conduction: a first parabolic example

The heat conduction in a 1D rod

- **Conservation of the internal energy** $\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} J_Q$ where $J_Q(t,x)$ is the heat flux across the section at x.
- The **entropy** $s = s(u)$ is chosen as **thermodynamic potential**. The intensive **effort** variable conjugated to the internal energy is then:

$$\frac{1}{T} = \frac{ds}{du}(u)$$

- The **driving force** is defined by

$$F' = -\frac{\partial}{\partial x}\frac{ds}{du}(u) = -\frac{1}{T^2}\frac{\partial T}{\partial x}$$

- **The heat flux** J_Q according to Fourier's law is then:

$$J_Q = \lambda(T,x)\, T^2\, F'$$

where $\lambda(T,x)$ denotes the heat conduction coefficient

The heat conduction ... continued

Canonical model for the heat equation

- The above relations may be grouped together:

$$\begin{bmatrix} -\frac{\partial u}{\partial t} \\ -F' \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} \frac{dS}{du} \\ J_Q \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e_\partial \\ f_\partial \end{bmatrix} = \begin{bmatrix} \frac{1}{T}|_\partial \\ J_Q|_\partial \end{bmatrix}$$

- The entropy balance equation reads now:

$$\frac{dS}{dt} = \frac{d}{dt}\int_\Omega s = \int_{\partial\Omega} \frac{J_Q}{T} + \int_\Omega \lambda(T,z)\, T^2\, F'^2$$

where $\frac{J_Q}{T}|_\partial$ is indeed the **entropy flow trough the boundary** and $\lambda T^2 F'^2$ the **irreversible entropy source** term .

A 3D parabolic example: the diffusion equation

[Baaiu et al. 2009] Mathematical and Computer Modelling of Dynamical Systems 22(1), 1-22

Mass conservation

$$\frac{d}{dt}\int_{\Omega} c \;=\; -\int_{\partial\Omega} J = -\int_{\Omega} \mathrm{div} J \quad \text{or in local form} \quad \frac{\partial c}{\partial t} = -div(J)$$

with c the molar density (concentration), J the molar flux, Ω the spatial domain and $\partial\Omega$ its closed boundary.

Thermodynamics phenomenological laws

$$\begin{aligned} \mu &= \mu(c) = \delta_c u(c) \quad &\textbf{internal energy function} \\ F &= -\mathrm{grad}\,(\mu) \quad &\textbf{driving force} \\ J &= J\,(F) \quad &\textbf{diffusion constitutive equation} \end{aligned}$$

where u denotes the internal energy (here the Gibbs free energy density), μ the chemical potential and J a phenomenological relation between the flux and the driving force

The diffusion equation ... continued

Canonical model for the diffusion equation

Although there is only one energy state space variable $c(x,t)dV$, mass conservation may be completed with the phenomenological diffusion relation:

$$\begin{bmatrix} -\frac{\partial c}{\partial t} \\ -F \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{bmatrix} \begin{bmatrix} \mu \\ J \end{bmatrix} \text{ and } \begin{bmatrix} e_\partial \\ f_\partial \end{bmatrix} = \begin{bmatrix} -\mu|_\partial \\ J|_\partial \end{bmatrix}$$

where $\mu|_\partial$ and $J|_\partial$ denotes respectively the chemical potential and the molar flux at the boundary $\partial\Omega$. The closure equations may then be written:

$$\begin{aligned} J &= J(F) \quad (\textit{e.g. Stefan-Maxwell or Fick relation for the diffusion}) \\ \mu(c) &= \delta_c \mathcal{H} \quad (\textit{e.g. Langmuir chemical potential}) \end{aligned}$$

The power balance reads

$$\frac{dH}{dt} = \int_{\partial\Omega} e_\partial \wedge f_\partial = \int_{\partial\Omega} \mu \wedge J$$

Conclusion: port-Hamiltonian formulation of distributed parameters systems

Conclusion on the port Hamiltonian formulation for DPS

Advertising for Dirac structures

- Hyperbolic and parabolic systems may be represented using the same canonical Stokes-Dirac structure
- Models are coordinate free and “causality-free”
- Many other interconnection structures are Dirac structures: serial, parallel, feedback, multi-scale, magneto-hydrodynamic, ...
- Dirac structure may be composed to represent complex systems
- there are many extension: external distributed ports, resistive ports, higher order (non canonical) Dirac interconnection structures

Conclusion ... continued

Other examples of boundary port Hamiltonian systems

The existing literature covers:

- Maxwell (3D) and telegraphist (1D) equations
- wave (1D), beam (1D) and plate/membrane (2D) equations
- ideal isentropic or Navier-Stokes fluids (3D)
- free surface SW (1D and 2D) Boussinesq (1D) or Korteweg - de Vries (1D) flows
- adsorption/diffusion (1D, 3D multi-scale), heat and reaction-diffusion equations (1D)
- magneto-hydrodynamics plasma flows (1D, 3D)
- piezzo-electric actuators
- . . .

Thank you for your attention